MODE CONVERSION OF GLOBAL MODES IN A UNIFORM CYLINDRICAL
MAGNETIZED PLASMA

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ABSTRACT

The propagation of electromagnetic waves in a uniform magnetized
plasma, bounded by a cylindrical conducting wall is studied using the
two-fluid theory, with the pressure term included. Taking the
cylindrical coordinates and perturbation of the form
\( \mathbf{E}(r) \exp(ikz-i\omega t) \) we obtain a sixth order equation for the
electric field component \( E_z \). Its solution is a sum of three
Bessel's or modified Bessel's functions. With appropriate boundary
conditions, we obtain the dispersion relation which is solved
numerically. The main result of this study is that the pressure term
causes the mode conversion of a backward wave to another backward
wave. The backward waves are modes which propagate in a frequency
range between the plasma and upper-hybrid frequencies.

1. INTRODUCTION

In this paper we study the global modes of electromagnetic
oscillation in a cylindrical waveguide. This is an important research
topic not only for practical purposes (fusion devices) but also for
basic research in electromagnetic theory [1]. We include the electron
temperature in the Trivelpiece and Gould problem [2,3], and our model
is not restricted to the slow wave cases \( \omega^2/k^2 \ll c^2 \) so that a
greater number of modes is analysed. One important result is that we
generalize the dispersion relation of Ghosh and Pal [4].

II. THE MODEL AND DISPERSION RELATION

In our model the plasma is treated as an adiabatic fluid in
which the ions are at rest (approximation valid in the high frequency
limit, \( \omega \gg \omega_p \) and \( \omega \gg \omega_i \)). We include a constant external
magnetic field, \( \mathbf{B}_0 \), along the waveguide. We apply a linearization
process in the form \( \mathbf{f}(\mathbf{r}) = f_0 + f(r) \exp(ikz-i\omega t) \), where \( k \)
is the wave number, \( n \) is an integer, \( \omega \) is the angular frequency of the
electromagnetic field, and where we utilize cylindrical coordinates
(the z axis of the coordinate system is the waveguide axis). In the
absence of an equilibrium electrostatic field, $\hat{E}_0 = 0$, and of an electron drift velocity, $\hat{u}_0 = 0$, the first order equations to be solved are (equations of continuity, of momentum transfer and Maxwell’s equations):

\[ i\omega p_1 = n_e m U^2 \nabla \cdot \hat{u}_1, \]

\[ i\omega n_0 m \hat{u}_1 = n_e c (\hat{E}_1 + \hat{u}_1 \times \hat{B}_3) + \nabla p_1, \]

\[ \nabla \times \hat{E}_1 = i\omega \mu_0 \hat{H}_1, \]

\[ \nabla \times \hat{H}_1 = -i\omega \varepsilon_0 \hat{E}_1 - n_e e \hat{u}_1, \]

where $p_1$, $n_0$, $m$, $U = \gamma k_B T_0 / m^{1/2}$, $\gamma$, $k_B$, $T_0$, $\hat{u}_1$, $\hat{e}$, $\hat{E}_1$, $\hat{H}_1$, $\mu_0$ and $\varepsilon_0$ are, respectively, the perturbed pressure, fluid density, electron mass, electron thermal velocity, ratio of specific heats (usually $\gamma = 5/3$), Boltzmann’s constant, electron temperature, perturbed fluid velocity, electron charge, perturbed electric and magnetic fields, vacuum magnetic permeability and vacuum dielectric constant. To obtain these equations we assumed also that the electron collision frequency is much smaller than the wave frequency $\omega$.

Assuming that $\hat{E}_0 = B_0 \hat{z}$ we get from these equations [5-7]:

\[ (\nabla_1^2 + b_1 \nabla_1 + b_2 \nabla_1 + b_3)E_z = (\nabla_1^2 + k_f^2)(\nabla_1^2 + k_s^2)(\nabla_1^2 + k_z^2)E_z = 0, \]

where

\[ b_1 = 2k_c^2 + k_s^2 - \frac{\omega^2}{\omega_z^2} k_c^2 - \frac{k_c^2}{k_s^2} \left( \frac{\omega^2}{\omega_z^2} k_f^2 - \frac{k_s^2}{k_f^2} \right), \]

\[ b_2 = k_c^2 + 2k_c^2 k_f^2 - \frac{\omega^2}{\omega_z^2} \left( k_c^2 + k_f^2 \right) \left( \frac{\omega^2}{\omega_z^2} k_f^2 - \frac{k_s^2}{k_f^2} \right), \]

\[ b_3 = k_c^2 k_f^2 - \frac{\omega^2}{\omega_z^2} \left( \frac{\omega^2}{\omega_z^2} k_f^2 - \frac{k_s^2}{k_f^2} \right), \]

and

\[ \omega_p = \left( \frac{n_e e^2}{\varepsilon_0 m} \right)^{\frac{1}{2}}, \omega_z = \frac{eB_0}{m}, \]

\[ k_f = \left( \frac{\omega^2 - \omega_z^2}{\omega_z^2} \right)^{\frac{1}{2}}, \quad k_s = \left( \frac{\omega^2 - \omega_z^2}{U^2} \right)^{\frac{1}{2}}, \]

\[ k_c = \left( \frac{\omega^2 - \omega_z^2}{c^2} \right)^{\frac{1}{2}}, \]

\[ \nabla_1^2 = \frac{d^2}{dt^2} + \frac{1}{r} \frac{d}{dr} - \frac{n_s^2}{r^2}, \]
and \( k_1, k_2, k_3 \) are analytic functions of \( b_1, b_2 \) and \( b_3 \), obtained by Cardan's formula [8]. All other field components can be obtained in term of \( E_z(r) \).

The solution of this equation is

\[
E_z = A_n J_n(rk_1) + B_n J_n(rk_2) + C_n J_n(rk_3).
\]

where \( J_n(x) \) is the \( n \)-th-order Bessel function of first kind. Applying the boundary conditions \( E_z(R) = E_{\theta}(R) = 0 \), and \( u_{\phi}(R) = 0 \), where \( R \) is the radius of the waveguide, we obtain the general dispersion relation given by [5-7]:

\[
n^2[F_1(L_2 - L_3) + F_2(L_3 - L_1) + F_3(L_1 - L_2)] + n[P_1(F_3 - F_2) - Q_1(L_3 - L_2)]
\]
\[
\times \frac{J'_n(Rk_1)}{J_n(Rk_1)} + n[P_2(F_1 - F_3) - Q_2(L_1 - L_3)] \frac{J'_n(Rk_2)}{J_n(Rk_2)}
\]
\[
+ n[P_3(F_2 - F_1) - Q_3(L_2 - L_1)] \frac{J'_n(Rk_3)}{J_n(rk_3)} + (Q_1P_2 - P_1Q_2) \frac{J'_n(Rk_1) J'_n(Rk_2)}{J_n(Rk_1) J_n(Rk_2)}
\]
\[
+ (Q_2P_3 - P_2Q_3) \frac{J'_n(Rk_2) J'_n(Rk_3)}{J_n(Rk_2) J_n(Rk_3)} + (Q_3P_1 - P_3Q_1) \frac{J'_n(Rk_3) J'_n(Rk_1)}{J_n(Rk_3) J_n(Rk_1)} = 0,
\]

where

\[
F_j = \frac{U^2[k_0^2(k_e^2 - k_j^2) + (k_+^2 - k_-^2)(k_-^2 - k_j^2)]}{Rk(c^2 - U^2)},
\]

\[
P_j = \frac{-k_j}{k\omega^2 \omega_p^2 G(c^2 - U^2)}[\omega^2 U^2(k_e^2 - k_j^2)(c^2 G + k_e^2 \omega_p^2)
\]
\[
+ k^2 \omega_p^2(c^2 - U^2)(\omega^2 k_e^2 - \omega_e^2 k_j^2) + U^2 \omega^2 \omega_p^2 (k_+^2 - k_-^2)(k_-^2 - k_j^2)],
\]

\[
L_j = \frac{U^2}{Rk \omega \omega_p^2 G(c^2 - U^2)}
\]
\[
\times [\omega_e^2(k_e^2 - k_j^2)(c^2 G + k_e^2 \omega_p^2) + c^2(\omega^2 k_e^2 - \omega_e^2 k_j^2)(k_+^2 - k_-^2)(k_-^2 - k_j^2)],
\]

\[
Q_j = \frac{-k_j}{c^2(c^2 - U^2) \omega_p^2 k \omega_e}
\]
\[
\times [k^2 \omega_e^4 \omega_p^4(c^2 - U^2) + \omega_e^2 k_j^2 U^2 \omega_p^2 c^2(k_e^2 - k_j^2) + c^4 U^2(\omega^2 k_e^2 - \omega_e^2 k_j^2)(k_+^2 - k_-^2)(k_-^2 - k_j^2)],
\]

and where \( j = 1, 2 \) or 3.
III. NUMERICAL RESULTS

In figure 1 we show the dispersion relation for a magnetized plasma waveguide with radius $R=0.085$ m, plasma frequency $\omega_p = 1.2 \times 10^{10}$s$^{-1}$, electron cyclotron frequency $\omega_c = 1.5 \times 10^{10}$s$^{-1}$, electron temperature $T_e = 40.$ eV and azimuthal wavenumber $n=1$. The figure shows the mode conversion pattern for values of the wave number $k$ around $1/R$. We also see the mode conversion for $kR \sim 4$. It is interesting to observe the occurrence of the mode conversion at a low temperature.

![Dispersion Relation](image)

REFERENCES