

A new method for inductance calculations

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Abstract. A new method for calculating inductances is presented. The method results in analytically simple expressions that replace the traditional tables and working formulae. Moreover, it overcomes the diverging self-inductances of thin wires.

1. Introduction

The concept of inductance arises naturally when studying the interaction energy between current-carrying circuits. This interaction energy has a factor that depends only on the geometry of the circuits. When we analyse the self-energy of a single closed circuit, this factor is called self-inductance; when we analyse the interaction energy of two distinct circuits, it is called mutual inductance.

With the theoretical development of electrodynamics, three main formulae appeared by which to calculate inductance: the expressions of Neumann, Weber and Maxwell [1, 2], and recently, a new one has been deduced from Graneau's work [3, p 212]. It is of great practical interest to have a method for calculating inductance with the desired degree of precision. This interest is of prominence mainly in technological areas, such as in the design of electrical circuits for communication.

There is a large literature about this topic. In general, tables and working formulae are presented using approximation methods like in [4, 5], or 'finite-current element analysis' like that of Graneau, [3, p 179] is applied. We shall present here a direct calculation of the integrals involved in the inductance formulae utilizing a more powerful method than those cited above. We compare our results with those obtained from the approximation methods and compare the four expressions for inductance calculations.

2. Inductance formulae

The four formulae cited above can be summarized in a single expression. Consider two current elements, $I_i d\mathbf{r}_i$ at \mathbf{r}_i and $I_j d\mathbf{r}_j$ at \mathbf{r}_j . The mutual inductance d^2M_{ij} between them can be written as

$$d^2M_{ij} = \frac{\mu_0}{4\pi} \left[\left(\frac{1+k}{2} \right) \frac{(d\mathbf{r}_i \cdot d\mathbf{r}_j)}{r_{ij}} + \left(\frac{1-k}{2} \right) \frac{(\hat{\mathbf{r}}_{ij} \cdot d\mathbf{r}_i)(\hat{\mathbf{r}}_{ij} \cdot d\mathbf{r}_j)}{r_{ij}} \right] \quad (1)$$

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where $\mu_0 \equiv 4\pi \times 10^{-7} \text{ kg m C}^{-2}$, $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$, $\hat{\mathbf{r}}_{ij} \equiv (\mathbf{r}_i - \mathbf{r}_j)/r_{ij}$ and k is a dimensionless constant. It is easy to see from (1) that $d^2M_{ij} = d^2M_{ji}$.

For $k = 1$ we obtain Neumann's formula $d^2M_{ij}^N$; with $k = -1$ there results Weber's formula $d^2M_{ij}^W$; for $k = 0$, Maxwell's formula $d^2M_{ij}^M$; and for $k = -5$ Graneau's formula $d^2M_{ij}^G$ [1–3]. Usually, the textbooks present only Neumann's formula, but it should be emphasized that beginning with Darwin's Lagrangian [6] we derive only Maxwell's formula.

It is known that, when we utilize (1) to calculate the mutual inductance between two closed circuits, the integrated value is independent of k [2]. This means that, in this case, the formulae of Neumann, Weber, Maxwell and Graneau yield the same result. This is due to the fact that these formulae differ by a total derivative whose contribution to a line integral around a closed contour is always zero. On the other hand, it is not yet known whether these formulae agree with one another for the self-inductance of a single closed circuit. In this work we compare these expressions for the self-inductance of the circuits of figures 1 and 2. This has never been done before.

The linear current element is an approximation of the reality. It is useful when the area of the cross section of the wire is small compared with its surface area (that is, when the diameter of the wire is much smaller than its length). When the linear elements are in contact, expression (1) cannot be utilized to calculate inductance due to problems of divergence. To solve this problem people usually utilize approximation methods. The first of these, the geometric-mean distance method of inductance calculations, was mainly developed by Maxwell [4]. Other methods utilizing approximations are described in [4, pp 6–28], namely the method of Taylor's series expansions and Lyle's method of equivalent filaments. The present work proposes a new method for calculating self-inductances. It allows, in principle, exact calculations for any geometry and can be utilized when the diameter of the wire is of the same order of magnitude as its length. Moreover, another advantage is that it yields algebraic expressions for the self-inductance. The other methods usually yield numerical values which need to be tabulated for different geometries.

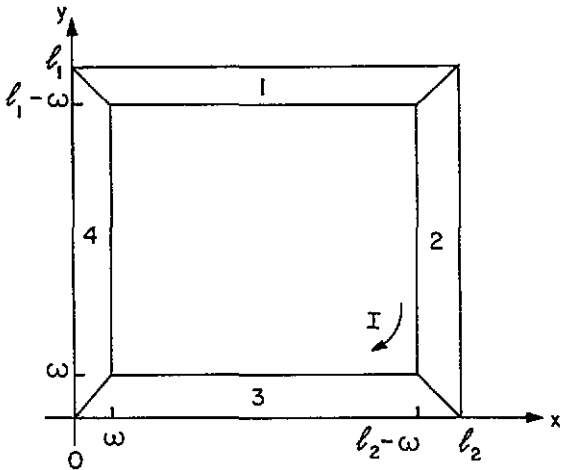


Figure 1. A rectangular circuit with sides l_1 and l_2 and a constant thickness w . There is a uniform and constant surface current I flowing in the circuit. In pieces 1 to 4 the direction of the current is given by, respectively, $+\hat{x}$, $-\hat{y}$, $-\hat{x}$ and $+\hat{y}$. Here \hat{x} and \hat{y} are the unit vectors along the x and y axes, respectively.

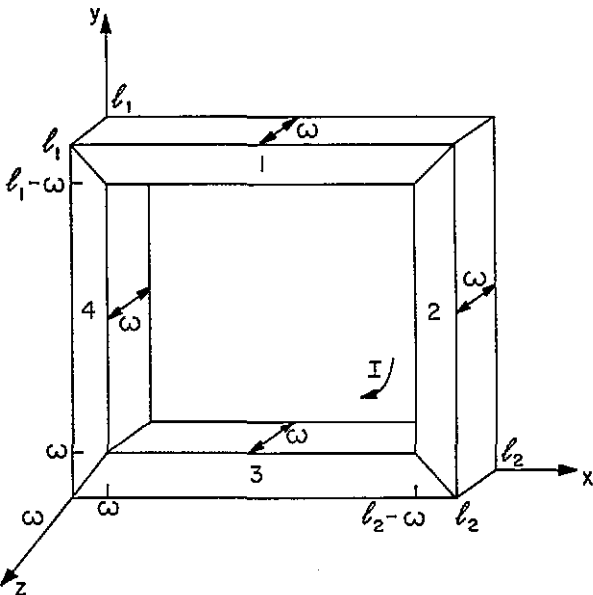


Figure 2. The same as in figure 1, but now with a uniform and constant volumetric current I flowing in a circuit with a square of side w as its cross section.

Our new method involves simply the substitution of the element $d\mathbf{r}$ by $\hat{\ell} dV/A$ in equation (1), where $\hat{\ell}$ is the unit vector indicating the direction of the current flow, A is the area of the cross section of the wire and dV an element of volume in the conductor. For two-dimensional current flow we replace $d\mathbf{r}$ by $\hat{\ell} da/\omega$, where ω is the width (transverse to $\hat{\ell}$) of the conductor and da an element of area in the conductor.

This idea is derived from Wesley's work [7]. He utilized for the first time a similar approach in order to compare the Ampère and Grassmann forces between current elements. Specifically, he replaced the current element $I d\mathbf{r}$ by $\mathbf{J} dV$, where \mathbf{J} is the volumetric current density vector (with units $A m^{-2}$). When the current flows uniformly in each cross section of the wire we have $\mathbf{J} = \hat{\ell} I/A$. On

utilizing this fact in Wesley's approach, $I d\mathbf{r} \rightarrow \mathbf{J} dV$ yields our result. Analogously, when the current flows only along a surface we have $I d\mathbf{r} \rightarrow \mathbf{K} da$, where \mathbf{K} is the surface current density vector (with units $A m^{-1}$). For uniform current flow along each section of the surface $\mathbf{K} = \hat{\ell} I/\omega$, yielding our previous result.

On making this substitution in (1), we obtain for surface and volumetric current elements, respectively

$$d^4 M_{ij} = \frac{\mu_0}{4\pi} \frac{1}{\omega_i \omega_j} \left[\left(\frac{1+k}{2} \right) \frac{(\hat{\ell}_i \cdot \hat{\ell}_j)}{r_{ij}} + \left(\frac{1-k}{2} \right) \frac{(\hat{r}_{ij} \cdot \hat{\ell}_i)(\hat{r}_{ij} \cdot \hat{\ell}_j)}{r_{ij}} \right] da_i da_j \quad (2)$$

$$d^6 M_{ij} = \frac{\mu_0}{4\pi} \frac{1}{A_i A_j} \left[\left(\frac{1+k}{2} \right) \frac{(\hat{\ell}_i \cdot \hat{\ell}_j)}{r_{ij}} + \left(\frac{1-k}{2} \right) \frac{(\hat{r}_{ij} \cdot \hat{\ell}_i)(\hat{r}_{ij} \cdot \hat{\ell}_j)}{r_{ij}} \right] dV_i dV_j \quad (3)$$

We use $\mathbf{K} da$ or $\mathbf{J} dV$ instead of $I d\mathbf{r}$, in some cases, because this makes the logarithmic divergence manageable. Usually, the opposite is done in order to reduce a higher dimensional integral to only one integration along a current line.

3. Calculation for specific configurations

We now calculate the self-inductance of the circuits of figures 1 and 2, beginning with some preliminary results.

3.1. Parallel straight wires

Let us calculate the mutual inductance between two parallel straight wires of length ℓ , separated by a distance h and carrying currents in opposite directions. Equation (1) with $d\mathbf{r}_i = dx_i \hat{x}$, $d\mathbf{r}_j = -dx_j \hat{x}$, $\mathbf{r}_i = x_i \hat{x}$ and $\mathbf{r}_j = x_j \hat{x} + h \hat{y}$ yields

$$M_{12} = -\frac{\mu_0}{4\pi} \int_0^\ell dx_i \int_0^\ell dx_j \left[\left(\frac{1+k}{2} \right) \frac{1}{[(x_i - x_j)^2 + h^2]^{1/2}} + \left(\frac{1-k}{2} \right) \frac{(x_i - x_j)^2}{[(x_i - x_j)^2 + h^2]^{3/2}} \right] \\ = -\frac{\mu_0}{4\pi} \left[2\ell \sinh^{-1} \left(\frac{\ell}{h} \right) + (3-k)[h - (\ell^2 + h^2)^{1/2}] \right] \quad (4)$$

Equation (4) depends on k and goes to infinity when $h/\ell \rightarrow 0$.

3.2. Rectangular surfaces

Consider a rectangular surface of sides ℓ and ω , with uniform current I in the direction of the side ℓ (figure 3). In this case we utilize equation (2) to calculate the self-inductance L of the rectangular current carrying surface. With $\hat{\ell}_i = \hat{\ell}_j = \hat{x}$, $\mathbf{r}_i = x_i \hat{x} + y_i \hat{y}$, $\mathbf{r}_j = x_j \hat{x} + y_j \hat{y}$, $\ell_i = \ell_j = \omega$, $da_i = dx_i dy_i$ and $da_j = dx_j dy_j$ we get

$$L = \frac{\mu_0}{4\pi} \frac{1}{\omega^2} \int_0^\ell dx_1 \int_0^\omega dy_1 \int_0^\ell dx_2 \int_0^\omega dy_2 \\ \times \left[\left(\frac{1+k}{2} \right) \frac{1}{[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}} \right]$$

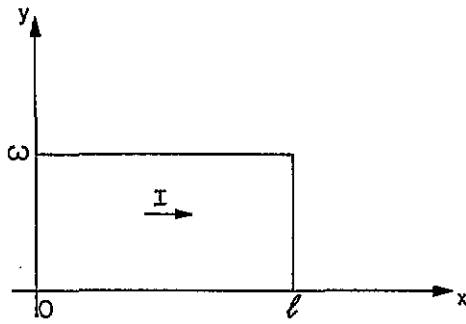


Figure 3. A rectangular surface with sides ℓ and ω with uniform current I flowing along its length.

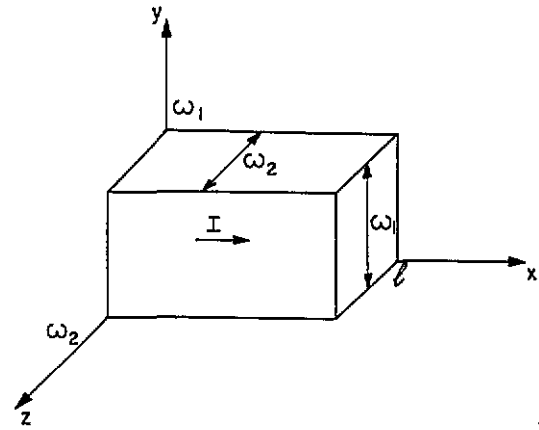


Figure 5. A wire of rectangular cross section with sides ω_1 and ω_2 , length ℓ and uniform current flow I .

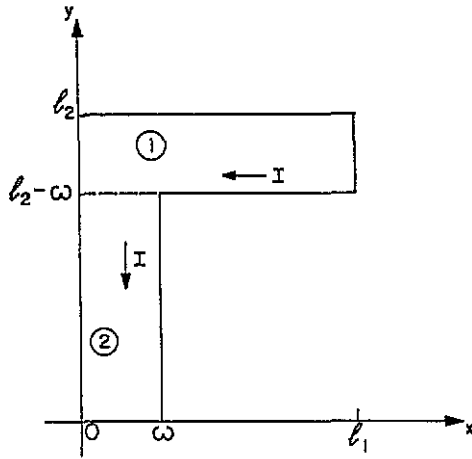


Figure 4. Rectangular surfaces in contact of lengths ℓ_1 and ℓ_2 and thickness ω , with uniform current flow I .

$$\begin{aligned}
 & + \left(\frac{1-k}{2} \right) \frac{(x_1 - x_2)^2}{[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{3/2}} \Big] \\
 = & \frac{\mu_0 \ell}{4\pi} \left[2 \sinh^{-1} \left(\frac{\ell}{\omega} \right) + (1+k) \left(\frac{\ell}{\omega} \right) \sinh^{-1} \left(\frac{\omega}{\ell} \right) \right. \\
 & \left. - \frac{(3-k)(\omega^2 + \ell^2)^{3/2}}{3\ell\omega^2} + (1-k) \frac{\ell}{\omega^2} (\omega^2 + \ell^2)^{1/2} \right. \\
 & \left. + \frac{2k}{3} \left(\frac{\ell}{\omega} \right)^2 + \frac{(3-k)}{3} \left(\frac{\omega}{\ell} \right) \right]. \tag{5}
 \end{aligned}$$

This is exact. It indicates how powerful this method of inductance calculations is. It yields exact results in analytical form. In order to compare it to the results of the literature, we consider $\omega \ll \ell$. Expanding the square brackets of (5) in ω/ℓ , and neglecting terms of order $(\omega/\ell)^3$ and above, yields

$$L \approx \frac{\mu_0 \ell}{4\pi} \left[2 \ln \left(\frac{2\ell}{\omega} \right) + k \right]. \tag{6}$$

This result goes to infinity when $\omega/\ell \rightarrow 0$, showing the divergence indicated above. As we are supposing that $\omega \ll \ell$, this approximate result is valid for any termination of the circuit and not only for a rectangular surface.

3.3. Rectangular surfaces in contact

We now calculate the mutual inductance between the rectangular surfaces 1 and 2 which are in contact, figure 4. Utilizing now $\hat{\ell}_i = -\hat{x}$, $\hat{\ell}_j = -\hat{y}$, $\mathbf{r}_i = x_i \hat{x} + y_i \hat{y}$,

$$\begin{aligned}
 \mathbf{r}_j &= x_j \hat{x} + y_j \hat{y}, \quad da_i = dx_i dy_i, \quad da_j = dx_j dy_j \quad \text{and} \\
 \ell_i &= \ell_j = \omega \quad \text{yields}
 \end{aligned}$$

$$\begin{aligned}
 M_{12} &= \frac{\mu_0}{4\pi} \frac{1}{\omega^2} \left(\frac{1-k}{2} \right) \int_0^{\ell_1} dx_i \int_{\ell_2-\omega}^{\ell_2} dy_i \int_0^{\omega} dx_j \int_0^{\ell_2-\omega} dy_j \\
 &\times \frac{(x_i - x_j)(y_i - y_j)}{[(x_i - x_j)^2 + (y_i - y_j)^2]^{3/2}}. \tag{7}
 \end{aligned}$$

We are interested only in the approximate result when $\omega \ll \ell_1$ and $\omega \ll \ell_2$. Neglecting terms of the orders $(\omega/\ell_1)^3$, $(\omega/\ell_2)^3$ and above yields

$$M_{12} \approx \frac{\mu_0}{4\pi} \left(\frac{1-k}{2} \right) [(\ell_1^2 + \ell_2^2)^{1/2} - \ell_1 - \ell_2]. \tag{8}$$

The last result does not diverge for any value of ℓ_1 or ℓ_2 , although the two pieces are in contact. Under this approximation ($\omega \ll \ell_1$ and $\omega \ll \ell_2$) the result is independent of ω , which is quite remarkable. Because the result (8) is valid only when $\omega \ll \ell_1$ and $\omega \ll \ell_2$, it will remain valid for any termination of the pieces.

3.4. Rectangular circuits

The expressions obtained in the last sections will be used to calculate the self-inductance L of the circuit of figure 1. In order to simplify the results, we consider that $\ell_1 \gg \omega$ and $\ell_2 \gg \omega$. The self-inductance of the circuit above can be written as (utilizing that, by symmetry, $L_1 = L_3$, $L_2 = L_4$, $M_{12} = M_{14} = M_{21} = M_{23} = M_{32} = M_{34} = M_{43} = M_{41}$, $M_{13} = M_{31}$ and $M_{24} = M_{42}$)

$$\begin{aligned}
 L &= \sum_{i=1}^4 L_i + \sum_{\substack{i,j=1 \\ i \neq j}}^4 M_{ij} \tag{9} \\
 &= 2L_1 + 2L_2 + 8M_{12} + 2M_{13} + 2M_{24}.
 \end{aligned}$$

We utilize equations (4) for M_{13} and M_{24} , equation (6) for L_1 and L_2 and equation (8) for M_{12} in expression (9). Neglecting terms of order $(\omega/\ell_1)^3$, $(\omega/\ell_2)^3$ and above yields the self-induction of this circuit as

$$L \approx \frac{\mu_0}{4\pi} \left[4\ell_2 \ln \left(\frac{2\ell_2}{\omega} \right) + 4\ell_1 \ln \left(\frac{2\ell_1}{\omega} \right) - 4\ell_2 \sinh^{-1} \left(\frac{\ell_2}{\ell_1} \right) - 4\ell_1 \sinh^{-1} \left(\frac{\ell_1}{\ell_2} \right) + 8(\ell_1^2 + \ell_2^2)^{1/2} - 2(\ell_1 + \ell_2) \right]. \quad (10)$$

The most important fact is that (10) is independent of k . Although the self-inductance of each piece and the mutual inductance of any two pieces depend on k , the same is not true for the self-inductance of this closed circuit. This is a completely non-trivial result.

We cannot yet compare this result to the literature because other workers usually tabulate the inductance of volumetric circuits. This is the subject of the next section.

3.5. Wires of rectangular cross section

Now, we calculate the self-inductance L of the wire of length ℓ and rectangular cross section of sides ω_1 and ω_2 . There is a uniform current I flowing along the wire, as indicated in figure 5.

With $\hat{\ell}_i = \hat{\ell}_j = \hat{x}$, $r_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$, $r_j = x_j\hat{x} + y_j\hat{y} + z_j\hat{z}$, $dV_i = dx_i dy_i dz_i$, $dV_j = dx_j dy_j dz_j$ and $A_i = A_j = \omega_1\omega_2$, we obtain with equation (3)

$$L = \frac{\mu_0}{4\pi} \frac{1}{\omega_1^2\omega_2^2} \times \int_0^\ell dx_1 \int_0^{\omega_1} dy_1 \int_0^{\omega_2} dz_1 \int_0^\ell dx_2 \int_0^{\omega_1} dy_2 \int_0^{\omega_2} dz_2 \times \left[\left(\frac{1+k}{2} \right) \frac{1}{[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2]^{1/2}} + \left(\frac{1-k}{2} \right) \frac{(x_1-x_2)^2}{[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2]^{3/2}} \right]. \quad (11)$$

Supposing (in order to compare to the tables) that $\omega_1 \ll \ell$ and $\omega_2 \ll \ell$ and neglecting terms of the order $(\omega_1/\ell)^3$, $(\omega_2/\ell)^3$ and above yields

$$L \approx \frac{\mu_0\ell}{4\pi} \left[2 \ln(2\ell) - \frac{\omega_1^2}{3\omega_2^2} \ln \omega_1 - \frac{\omega_2^2}{3\omega_1^2} \ln \omega_2 - \left(1 - \frac{\omega_2^2}{6\omega_1^2} - \frac{\omega_1^2}{6\omega_2^2} \right) \ln(\omega_1^2 + \omega_2^2) - \frac{4}{3} \frac{\omega_1}{\omega_2} \times \arctan \left(\frac{\omega_2}{\omega_1} \right) - \frac{4}{3} \frac{\omega_2}{\omega_1} \arctan \left(\frac{\omega_1}{\omega_2} \right) + \frac{7}{6} + k \right]. \quad (12)$$

For $\omega_1 = \omega_2 \equiv \omega$ this reduces to

$$L \approx \frac{\mu_0\ell}{4\pi} \left[2 \ln \left(\frac{2\ell}{\omega} \right) - \frac{2}{3} \ln 2 - \frac{2\pi}{3} + \frac{7}{6} + k \right]. \quad (13)$$

This result diverges when $\omega/\ell \rightarrow 0$. Because (13) is valid for $\omega \ll \ell$, it will remain valid for any termination of the piece.

These two last results can be compared to the literature. In all tables only Neumann's formula ($k = 1$) is presented. The value of $(4\pi/\mu_0)(L/\ell)$ is a dimensionless parameter. We will take this parameter with three significant digits.

Grover, for instance, gives the following expressions for $\omega_2 = 2\omega_1 \equiv 2\omega$ and for $\omega_1 = \omega_2 \equiv \omega$ [4, p 35], respectively,

$$\frac{4\pi L}{\mu_0\ell} \approx 2 \ln \left(\frac{\ell}{\omega} \right) + 0.185 \quad (14)$$

$$\frac{4\pi L}{\mu_0\ell} \approx 2 \ln \left(\frac{\ell}{\omega} \right) + 0.996. \quad (15)$$

Our formulae yield the following algebraic results, with $k = 1$:

$$\frac{4\pi L}{\mu_0\ell} \approx 2 \ln \left(\frac{\ell}{\omega} \right) + \frac{2}{3} \ln 2 - \frac{7}{24} \ln 5 + 2 \arctan 2 - \frac{4\pi}{3} + \frac{13}{6} \quad (16)$$

$$\frac{4\pi L}{\mu_0\ell} \approx 2 \ln \left(\frac{\ell}{\omega} \right) + \frac{4}{3} \ln 2 - \frac{2\pi}{3} + \frac{13}{6}. \quad (17)$$

The values of the numerical constants (to three significant digits) in (16) and (17) are given by, respectively, 0.185 and 0.996. This shows the correctness of our approach, and how it can be widely applied in other geometries.

We now continue our analysis to calculate the self-inductance of the circuit of figure 2. What remains to be calculated is the mutual inductance between opposite sides (such as sides 1 and 4), and between adjacent sides (such as sides 1 and 2). Because we are considering $\omega_1 \ll \ell_1$, $\omega_2 \ll \ell_1$, $\omega_1 \ll \ell_2$ and $\omega_2 \ll \ell_2$, result (4) will be the approximate value of the mutual inductance between opposite sides. Analogously, the mutual inductance between any two adjacent volumetric sides will be given by (8).

On utilizing in equation (9) the expressions (13) for L_1 and L_2 , (8) for M_{12} and (4) for M_{13} and M_{24} we get, with $\omega_1 = \omega_2 \equiv \omega$ (neglecting terms of order $(\omega/\ell_1)^3$, $(\omega/\ell_2)^3$ and above)

$$L \approx \frac{\mu_0}{4\pi} \left[4\ell_2 \ln \left(\frac{2\ell_2}{\omega} \right) + 4\ell_1 \ln \left(\frac{2\ell_1}{\omega} \right) - 4\ell_2 \sinh^{-1} \left(\frac{\ell_2}{\ell_1} \right) - 4\ell_1 \sinh^{-1} \left(\frac{\ell_1}{\ell_2} \right) + 8(\ell_1^2 + \ell_2^2)^{1/2} + 2(\ell_1 + \ell_2) \times \left(\frac{1}{6} - \frac{2}{3} \ln 2 - \frac{2\pi}{3} \right) \right]. \quad (18)$$

The self-inductance for the circuit of figure 2 is independent of k , as was the case with the circuit of figure 1. Let us compare it to Grover's tabulated result for this geometry. In equation (9) with Grover's value (15) for L_1 and L_2 , (8) ($k = 1$) for M_{12} and (4) ($k = 1$) for M_{13} and M_{24} , we get his result as

$$L \approx \frac{\mu_0}{4\pi} \left[4\ell_2 \ln \left(\frac{2\ell_2}{\omega} \right) + 4\ell_1 \ln \left(\frac{2\ell_1}{\omega} \right) - 4\ell_2 \sinh^{-1} \left(\frac{\ell_2}{\ell_1} \right) - 4\ell_1 \sinh^{-1} \left(\frac{\ell_1}{\ell_2} \right) + 8(\ell_1^2 + \ell_2^2)^{1/2} + 2(\ell_1 + \ell_2) \times (-2 \ln 2 - 1.004) \right]. \quad (19)$$

On comparing (18) and (19) we see that they may differ only in the numerical coefficient of the factor $2(\ell_1 + \ell_2)$.

In equation (18) we have: $[1/6 - 2/3(\ln 2) - (2\pi)/3] \approx -2.390$; and by the approximation method used in Grover's book, $(-2 \ln 2 - 1.004) \approx -2.390$. Consequently, there is an excellent agreement between our method and the approximation utilized in Grover's book.

4. Conclusions

We have presented a new way of making inductance calculations. One of the advantages is that we can obtain, in principle, exact and algebraic results like the ones obtained in (4) and (5). Our results for Neumann's formula were compared to those obtained by Grover and the agreement was excellent. For the configurations utilized in this work we have obtained that the self-inductance of each piece, or the mutual inductance of any two pieces, depends on k . So, the inductance has different values for the formulae of Neumann, Weber, Maxwell and Graneau. Despite this fact, the self-inductances of the closed circuits of figures 1 and 2 are independent of k and have the same value for all these formulae.

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References

- [1] Assis A K T 1994 *Weber's Electrodynamics* (Dordrecht: Kluwer) pp 98–102
- [2] Whittaker E T 1973 *A History of the Theories of Aether and Electricity* vol I (New York: Humanities Press) p 233
- [3] Graneau P 1985 *Ampère–Neumann Electrodynamics of Metals* (Nonantum: Hadronic)
- [4] Grover F W 1946 *Inductance Calculations – Working Formulas and Tables* (New York: Van Nostrand)
- [5] Knowlton A E 1949 *Standard Handbook for Electrical Engineers* (New York: McGraw-Hill)
- [6] Jackson J D 1975 *Classical Electrodynamics* 2nd edn (New York: Wiley) pp 593–5
- [7] Wesley J P 1990 *Found. Phys. Lett.* 3 443