Consequences of a generalized law of the lever

A. K. T. Assis[a)] and F. M. d. M. Ravanelli[b)]
Institute of Physics "Gleb Wataghin," University of Campinas—UNICAMP, 13083-970 Campinas, São Paulo, Brazil

(Received 29 April 2008; accepted 14 August 2008)

We discuss the controversy about the demonstration of the law of the lever as given by Archimedes. One aspect of the discussion concentrates on the meaning of the postulates which he utilized. We analyze what consequences would arise if nature behaved in such a way that the lever followed a generalized power law. In particular, we consider the cases of a torque independent of the distances of the bodies to the fulcrum, proportional to these distances, and quadratic in the distances.

[DOI: 10.1119/1.2978002]

I. INTRODUCTION

Archimedes (287–212 BCE) demonstrated the law of the lever in propositions 6 and 7 of his work *On the Equilibrium of Planes*, or *Centres of Gravity of Planes*:[1] These propositions are

> Proposition 6: Commensurable magnitudes are in equilibrium at distances reciprocally proportional to the weights.

> Proposition 7: However, even if the magnitudes are incommensurable, they will be in equilibrium at distances reciprocally proportional to the magnitudes.

Heath combined these two propositions in his paraphrase of Archimedes's work:[2] "Propositions 6, 7. Two magnitudes, whether commensurable [Prop. 6] or incommensurable [Prop. 7], balance at distances reciprocally proportional to the magnitudes."

Suppose we have weights W_A and W_B on two sides of a lever supported by their centers of gravity located at distances d_A and d_B from the fulcrum F. According to the law of the lever, equilibrium will prevail if

$$\frac{W_A}{W_B} = \frac{d_B}{d_A}. \tag{1}$$

In Fig. 1 we present a lever in equilibrium (that is, at rest relative to the ground, although free to rotate around the fulcrum F), with bodies A and B on opposite sides of the fulcrum. We will assume weightless beams and weightless strings connecting the bodies to the lever.

Archimedes also demonstrated how to locate the center of gravity of a triangle:[3]

> Proposition 13: In any triangle the center of gravity lies on the straight line joining any vertex to the middle point of the base.

> Proposition 14: In any triangle the center of gravity is the point in which the straight lines of the triangle joining the vertices to the middle points of the sides meet.

The straight line connecting each vertex to the middle point of the opposite side of a triangle is called the median. The three medians meet at a single point which is the center of gravity, G, of the triangle. This point is also called the barycenter or centroid of the triangle and is represented in Fig. 2. The center of gravity is always inside the triangle and has an important property. The distance from the vertex to the center of gravity is always twice the distance from the center of gravity to the midpoint of the opposite side. In Fig. 2 the midpoint of the side B_1C_1 is represented by the point A_2. This property means that $A_1G=2GA_2$ or $GA_2=A_1A_2/3$. Analogous relations are obtained for the other medians.

Another property of the center of gravity of any body is that it is a single point. Suppose that the body can rotate around an axis. If the body is released from rest, it will remain in equilibrium whenever this axis passes through the center of gravity of the body. Suppose, for instance, that the triangle is released from rest in a horizontal plane in such a way that it can rotate around a horizontal axis passing through it. It is found that it will remain in equilibrium for all axes that pass through its center of gravity G.

To demonstrate these results Archimedes utilized seven postulates:[4]

(1) We postulate that equal weights at equal distances are in equilibrium, and that equal weights at unequal distances are not in equilibrium, but incline towards the weight which is at the greater distance.
(2) That if, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium, but incline towards that weight to which something has been added.
(3) Similarly that, if anything be taken away from one of the weights, they are not in equilibrium, but incline towards that weight from which nothing has been taken away.
(4) When equal and similar figures are made to coincide, their centers of gravity likewise coincide.
(5) In figures which are unequal, but similar, the centers of gravity will be similarly situated. We say that points are similarly situated in relation to similar figures if straight lines drawn from these points to the equal angles make equal angles with the homologous sides.
(6) If magnitudes at certain distances be in equilibrium, other [magnitudes] equal to them will also be in equilibrium at the same distances.
(7) In any figure whose perimeter is concave in the same direction the center of gravity must be within the figure.

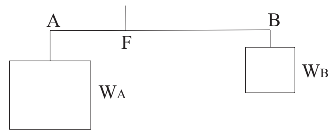

Fig. 1. Lever in equilibrium relative to the ground.

Although the concept of the center of gravity appears in postulate 4, it is not defined in any extant work of Archimedes. Heath, Duhem, Stein, Dijksterhuis, Assis, and many others have studied how Archimedes implicitly utilized this concept to calculate the center of gravity of many figures (such as a triangle and a parabola).[5–10] They also studied earlier authors such as Heron (first century CE), Eutocius (450–540 CE), and Pappus (fourth century CE) who had access to other works of Archimedes no longer extant. From these studies it seems that Archimedes understood the center of gravity to be a point such that if the body were suspended from that point, released from rest and free to rotate in all directions around that point, the body would remain at rest and would preserve its original position no matter what the initial orientation of the body relative to the ground.

Archimedes's demonstration of the law of the lever was criticized by Ernst Mach.[11] He quoted only the first two postulates and concluded (italics in original):

> …the assumption that the equilibrium-disturbing effect of a weight P at the distance L from the axis of rotation is measured by the product $P \cdot L$ (the so-called statical moment), is more or less covertly or tacitly introduced by Archimedes and all his successors.
>
> First, it is obvious that if the arrangement is absolutely symmetrical in every respect, equilibrium obtains on the assumption of *any* form of dependence whatever the disturbing factor on L, or generally, on the assumption $P \cdot f(L)$; and that consequently the *particular* form of dependence PL cannot possibly be inferred from the equilibrium. The fallacy of the deduction must accordingly be sought in the transformation to which the arrangement is subjected. Archimedes makes the action of two equal weights to be the same under all circumstances as that of the combined weights acting at the middle point of their line of junction.

Stein and Dijksterhuis objected to Mach's criticism.[8–12] In particular, they pointed out that in his demonstration of the law of the lever Archimedes utilized not only the first two postulates, as mentioned by Mach, but also his sixth postulate. Stein and Dijksterhuis understood Archimedes to interpret "magnitudes equal to other magnitudes" as "magnitudes of the same weight" and "magnitudes at the same distances" as "magnitudes the centers of gravity of which lie at the same distances from the fulcrum." This interpretation conferred a reasonable meaning to the sixth postulate and removed Mach's objections to Archimedes's demonstration of the law of the lever.[13]

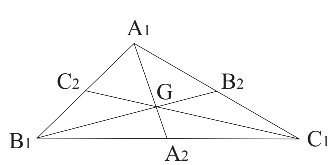

Fig. 2. Center of gravity of a triangle.

We illustrate this interpretation of the sixth postulate with two examples. Suppose that N bodies of weights $W_1, W_2, \ldots, W_N$ are suspended by their centers of gravity at the distances $d_1, d_2, \ldots, d_N$ from the fulcrum of a lever in equilibrium. In the first example we replace body 2 by another body M suspended by its center of gravity at the same distance d_2 from the fulcrum. According to this interpretation of the sixth postulate, equilibrium will remain provided $W_M = W_2$. As a less trivial example we consider the same N bodies, but in a situation for which $W_1 = W_2 \equiv W$ and that these two bodies are suspended at distances d_1 and d_2 from the same side of the fulcrum. Archimedes demonstrated in the fourth proposition that "if two equal magnitudes have not the same center of gravity, the center of gravity of the magnitude composed of the two magnitudes will be the middle point of the straight line joining the centers of gravity of the magnitudes."[14] We now replace bodies 1 and 2 by a single body M of weight $W_M = 2W$. According to this interpretation of the sixth postulate, equilibrium will not be disturbed if it is suspended at the joint center of gravity of bodies 1 and 2; that is, at a distance $(d_1 + d_2)/2$ from the same side of the fulcrum. In other words, Stein and Dijksterhuis mean that we may always replace two weights by their total weight acting at their center of gravity without disturbing the equilibrium of a lever.[8,12]

We agree with Stein and Dijksterhuis's point of view. But their interpretation of postulate 6 requires a serious emendation of the Greek text of this postulate, because it contains nothing about centers of gravity. To illustrate the crucial role played by postulate 6 in Archimedes's demonstration of the law of the lever, we consider what would be the consequences if nature behaved in such a way that the law of the lever were quadratic in the distances of the bodies or independent of these distances.

II. A GENERALIZED LAW OF THE LEVER

Suppose a horizontal beam acts as a lever that can rotate around a horizontal axis orthogonal to the beam of the lever passing through its fulcrum. We consider N bodies on one side of the fulcrum and M bodies on the other side. A generic body i has weight W_i, with its center of gravity being suspended by the beam of the lever at a distance d_i from the fulcrum. We define a generic "alpha" torque τ exerted by these bodies as $\tau_N \equiv \sum_{i=1}^{N} W_i (d_i)^\alpha$ and $\tau_M \equiv \sum_{i=N+1}^{M} W_i (d_i)^\alpha$. The exponent α characterizes the behavior of the lever as a function of the distance to the fulcrum. In real life $\alpha = 1$. In this work we wish to compare this normal condition with hypothetical situations for which $\alpha \neq 1$. To this end we postulate the following behavior for the lever released at rest horizontally, being free to rotate around the fulcrum:

$$\text{If } \tau_N = \tau_M, \quad \text{the lever remains in equilibrium}.$$

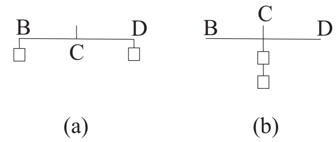

Fig. 3. A lever supporting two equal weights can rotate about C. (a) It remains in equilibrium for any value of α when the weights are suspended at B and D with $BC=CD$, or (b) when they are suspended at C.

$$\text{If } \tau_N > \tau_M, \quad \text{the set of } N \text{ bodies inclines toward the ground.}$$
$$\text{If } \tau_N < \tau_M, \quad \text{the set of } M \text{ bodies inclines toward the ground.} \tag{2}$$

Our proposal is not original. Mach had already suggested a generic torque proportional to the weight of the bodies and to a function $f(L)$ of their distances to the fulcrum.[11] Czwalina considered a quadratic law of the lever in his analysis of Archimedes's work.[15] The main point of our paper is to explore what would happen to the logical structure of the proof of the law of the lever and of the center of gravity of a triangle if the torque followed a power law. The goal is to separate the parts of the proof that are logically independent of one another.

We now consider simple symmetrical situations of equilibrium. In Fig. 3(a) two equal weights W are suspended at points B and D from a lever which can rotate around a fulcrum located at C. If $BC=CD$, the lever will remain in equilibrium for all values of α. The lever will also remain in equilibrium for any value of α when the two weights are suspended at C, as in Fig. 3(b). That is, in this case we can replace the two equal weights at B and D by a single body of twice the weight at the midpoint C without disturbing the equilibrium of the lever for any value of α. The center of gravity of the two equal weights W_B and W_D can be considered their midpoint. As we have seen, Archimedes proved this fact in proposition 4.

Now let us see how Archimedes demonstrated the law of the lever considering the simplest case of Fig. 4. The propositions must follow from the postulates. However, in his demonstration of the propositions Archimedes did not explicitly mention any of his postulates. But by following his reasoning we can understand how he utilized these postulates implicitly. Consider three equal weights suspended at points A, B, and D. The lever is free to rotate around the middle point B. If $AB=BD$, the lever will remain in equilibrium [see Fig. 4(a)]. As we have seen, the center of gravity of the two equal weights W_B and W_D is in the middle point of the straight line connecting their centers of gravity. And C is the midpoint of the segment BD. By postulate 6 we will not disturb the equilibrium of the lever by replacing bodies B and D by a single body of twice the weight acting at C. We then arrive at the situation of Fig. 4(b). This configuration is a special case of the law of the lever because $W_A/W_C=BC/AB=1/2$, or $BC=\frac{1}{2}AB$ in agreement with Eq. (1). In essence, what allows Archimedes to conclude that the situation shown in Fig. 4(b) will remain in equilibrium beginning from the symmetrical equilibrium of Fig. 4(a) is postulate 6.

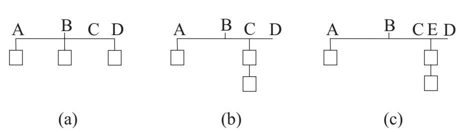

Fig. 4. A lever supporting three equal weights can rotate about B. (a) The lever remains in equilibrium if $AB=BD$. (b) For $\alpha=1$ it remains in equilibrium with two equal weights joined at C such that $BC=0.5AB$. (c) For $\alpha=2$ it remains in equilibrium with two equal weights joined at E such that $BE=(\sqrt{2}/2)AB\approx 0.707AB$.

Let us assume $\alpha\neq 1$ and the postulates given by Eq. (2). In this case the situation of Fig. 4(a) continues to be an equilibrium configuration, but the situation in Fig. 4(b) is no longer in equilibrium. If $\alpha<1$, the weights at C will incline toward the ground. In contrast, if $\alpha>1$, the weight A will incline toward the ground. The new equilibrium situation according to Eq. (2) and the definition of the torque is the configuration with the equal weights W_B and W_D acting together at another point E such that $W_A/W_E=(BE/AB)^\alpha$, that is, $BE=(1/2)^{1/\alpha}AB$. If $\alpha=2$, $BE=(\sqrt{2}/2)AB\approx 0.707AB$ as shown in Fig. 4(c). If $\alpha=0$, the solution diverges; if $\alpha=1/2$, we have $BE=\frac{1}{4}AB$.

We can go from the configuration of Fig. 3(a) to that of Fig. 3(b) without disturbing the equilibrium of this lever for all values of α. In contrast, we can go from the configuration of Fig. 4(a) to that of Fig. 4(b) without disturbing the equilibrium of this lever only if $\alpha=1$. If $\alpha=2$, we can maintain the equilibrium of the lever of Fig. 4(a) only by combining the equal weights W_B and W_D at another point E given by $BE=\sqrt{2}AB/2\approx 0.707AB$ [see Fig. 4(c)]. This last situation shows that Archimedes's postulate 6, as interpreted by Stein and Dijksterhuis, would not be valid if $\alpha=2$.[8,12] This conclusion lends support to their interpretation of this postulate and to the fact that this postulate was essential to allowing Archimedes to demonstrate the law of the lever.

Czwalina discussed Archimedes's proposition 6 and claimed that his demonstration is based on a fallacy.[15] His criticism is founded on the fact that the first five propositions of Archimedes would remain valid even for a quadratic law of the lever, which is not the case for proposition 6. He concluded that proposition 6 is not a logical consequence of the previous propositions. However, Czwalina, like Mach before him, did not notice the relevance of the sixth postulate for the demonstration of proposition 6. According to Stein and Dijksterhuis, proposition 6 is a logical consequence of the first five propositions, combined with postulate 6.[8,16] Accepting this postulate with the previous interpretation removes any supposed fault in Archimedes's proof of the law of the lever.

Mach still has a point for if we emend postulate 6 as proposed by Stein and Dijksterhuis, the formal structure of the proof may be rescued.[8,12] But postulate 6 then implicitly contains the linear law of the lever. However, as Archimedes explicitly assumed postulate 6 as an axiom, this assumption removes any circularity in his proof.

The sixth postulate is utilized by Archimedes not only to arrive at the law of the lever, but also to calculate the center of gravity of a triangle, and thus it is a powerful postulate from which novel results can be demonstrated. We discuss this subject in the next section.

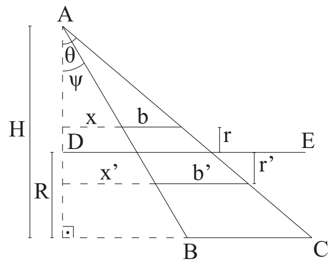

Fig. 5. A horizontal triangle ABC that can rotate around the axis DE.

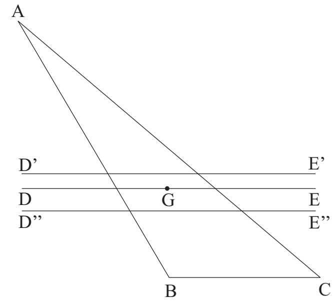

Fig. 6. The horizontal triangle ABC has G as the intersection of the medians. If $\alpha=1$, the equilibrium axis parallel to BC is DE, passing through G. If $\alpha=2$, the equilibrium axis is $D'E'$ closer to the vertex A; if $\alpha=0$, the equilibrium axis is $D''E''$ closer to BC.

III. EQUILIBRIUM OF A TRIANGLE

We consider the generic horizontal triangle ABC of Fig. 5 with height H and base BC. This triangle can rotate freely around the axis DE which is fixed relative to the ground and is parallel to BC. We want to find the distance R between this axis and the side BC that will let this triangle be in equilibrium for a given value of α, with $0<R<H$. We call r the distance between this axis and a strip of the triangle with length b and thickness dr on one side of the axis, and r' the analogous distance of a strip $b'dr'$ on the other side.

From the angles θ and ψ shown in Fig. 5 we have

$$\tan\theta = \frac{b+x}{H-R-r} = \frac{b'+x'}{H-R+r'}, \tag{3}$$

$$\tan\psi = \frac{x}{H-R-r} = \frac{x'}{H-R+r'}. \tag{4}$$

According to the assumption in Eq. (2) and the previous definition of the torque, equilibrium will occur when $\int r^\alpha dW$ integrated over one side of the axis is equal to $\int r'^\alpha dW'$ integrated over the other side, where dW and dW' are the weight of the strips bdr and $b'dr'$, respectively. For a homogeneous triangle we have $dW/W=da/A$, where $da=bdr$ is the area of the strip corresponding to the weight dW, W is the total weight of the triangle, and A is its total area. The equilibrium condition implies

$$\int_{r=0}^{H-R} r^\alpha b dr = \int_{r'=0}^{R} r'^\alpha b' dr'. \tag{5}$$

We utilize Eqs. (3) and (4) and obtain

$$b = (\tan\theta - \tan\psi)(H-R-r), \tag{6}$$

$$b' = (\tan\theta - \tan\psi)(H-R+r'). \tag{7}$$

We define the constant k such that $H-R\equiv kR$, or $R=H/(1+k)$. Because we wish to have $0<R<H$, we must have $0<k<\infty$. Equating the torques of both sides of the fulcrum yields

$$\int_0^{kR} r^\alpha (kR-r) dr = \int_0^R r'^\alpha (kR+r') dr'. \tag{8}$$

That is,

$$\left[\frac{kR(kR)^{\alpha+1}}{\alpha+1} - \frac{(kR)^{\alpha+2}}{\alpha+2}\right] = \left[\frac{kR(R)^{\alpha+1}}{\alpha+1} + \frac{R^{\alpha+2}}{\alpha+2}\right]. \tag{9}$$

This last expression yields the following equation for k,

$$k^{\alpha+2} - (\alpha+2)k - (\alpha+1) = 0. \tag{10}$$

If $\alpha=1$, Eq. (10) reduces to $k^3-3k-2=0$. There are three roots, $k_1=2$, $k_2=-1$, and $k_3=-1$. Only the first solution is physically reasonable, implying $R=H/3$. This solution is the usual one of an axis passing through the center of gravity of the triangle; that is, passing through the intersection of the medians. To demonstrate this result Archimedes also implicitly utilized postulate 6. For a discussion see Refs. 17 and 18. Archimedes divided the whole triangle into four smaller and congruent triangles. He then combined two of these triangles into a single body with twice the weight of one of these small triangles and acting at their combined center of gravity. According to the sixth postulate the triangle would remain in equilibrium after this replacement. His demonstration is then correctly made, although the sixth postulate was utilized only implicitly by Archimedes.

For $\alpha=2$, Eq. (10) reduces to $k^4-4k-3=0$ which has roots $k_1\approx -0.693$, $k_2\approx -0.546-1.459i$, $k_3\approx -0.546+1.459i$, and $k_4\approx 1.784$. Only the fourth solution is compatible with the condition $0<R<H$. We are then led to $R\approx H/2.784 \approx 0.359H$. This solution means that this axis parallel to the side BC will not pass through the intersection of the medians, but will be closer to the vertex A.

If $\alpha=0$, Eq. (10) reduces to $k^2-2k-1=0$ which has roots $k=1+\sqrt{2}\approx 2.414$ and $k=1-\sqrt{2}\approx -0.414$. Only the first root is compatible with the condition $0<R<H$, and hence $R \approx H/3.414 \approx 0.293H$. This solution means that this axis parallel to the side BC will not pass through the intersection of the medians and will be closer to the base BC. The axis is the straight line parallel to the base which divides the triangle ABC into two figures of equal area and equal weight, namely, the smaller triangle at the top and the trapezoid at the bottom.

In Fig. 6 we show the location of the axes parallel to BC such that the horizontal triangle is in equilibrium after being released from rest. If $\alpha=1$, this axis is DE which passes through the intersection of the medians, that is, $R=H/3 \approx 0.333H$. If $\alpha=2$, an axis passing through the intersection of the medians will not let the triangle be in equilibrium, and the vertex A will incline toward the ground. In this case the triangle will remain in equilibrium for an axis $D'E'$ parallel to BC located at a the distance $R\approx 0.359H$. This equilibrium axis for $\alpha=2$ is closer to the vertex A than the equilibrium axis for $\alpha=1$. If $\alpha=0$, the triangle will remain in equilibrium for an axis $D''E''$ parallel to BC located at the distance $R \approx 0.293H$. This axis divides the triangle into two regions of equal areas.

Similar results will be obtained for axes parallel to the sides AB and AC. In particular, the three equilibrium axes parallel to the three sides of a triangle will meet at the usual center of gravity G of the triangle only if $\alpha=1$. If $\alpha=2$ or $\alpha=0$, the three axes will not meet at a single point and none of them will pass through G.

This conclusion shows again that postulate 6 is essential to demonstrate the usual results related to the center of gravity of bodies of extended spatial extent.

ACKNOWLEDGMENTS

The authors thank the referee for his/her constructive remarks on the first version of this paper. One of the authors (FMdMR) thanks PIBIC/SAE/UNICAMP for an undergraduate research fellowship during which this work was completed.

[a] Electronic mail: assis@ifi.unicamp.br
[b] Electronic mail: famatos@ifi.unicamp.br

[1] E. J. Dijksterhuis, *Archimedes* (Princeton U. P., Princeton, 1987), pp. 289 and 305. Translated by C. Dikshoorn.
[2] Archimedes, *The Works of Archimedes* (Dover, New York, 2002), p. 192. Translated and edited by T. L. Heath.
[3] Reference 1, pp. 309 and 311.
[4] Reference 1, pp. 286–287.
[5] T. Heath, *A History of Greek Mathematics, Vol. II: From Aristarchus to Diophantus* (Clarendon, Oxford, 1921), pp. 24, 302, 350–351, and 430.
[6] Reference 2, pp. clxxxi–clxxxii.
[7] P. Duhem, *The Origins of Statics* (Kluwer, Dordrecht, 1991), pp. 264–267, 457–465, and 307. Translated by G. F. Leneaux, V. N. Vagliente, and G. H. Wagener.
[8] W. Stein, "Der Begriff des Schwerpunktes bei Archimedes," Quellen und Studien zur Geschichte der Mathematik, Physik und Astronomie. Abt. B: Quellen **I**, 221–244 (1930).
[9] Reference 1, pp. 17, 47–48, 289–304, 315–316, 321–322, and 435–436.
[10] A. K. T. Assis, *Archimedes, the Center of Gravity, and the First Law of Mechanics* (Apeiron, Montreal, 2008), pp. 72 and 97–105. Available at ⟨www.ifi.unicamp.br/~assis⟩.
[11] E. Mach, *The Science of Mechanics—A Critical and Historical Account of Its Development*, 6th ed. (Open Court, La Salle, 1960), pp. 19–20. Translated by J. McCormack.
[12] Reference 1, pp. 289–304.
[13] Reference 10, Sec. 9.7, pp. 177–185.
[14] Reference 1, p. 288.
[15] Archimedes, *Die Quadratur der Parabel und über das Gleichgewicht ebener Flächen oder über den Schwerpunkt ebener Flächen* (Akad. Verlagsgesellsch, Leipzig, 1923). Vol. 203 of Ostwald's Klassiker der exakten Wissenschaften. Übers. u. mit Anmerkungen vers. v. A. Czwalina-Allenstein.
[16] Reference 1, p. 294.
[17] Reference 1, p. 311.
[18] Reference 10, pp. 182–185.

PARALLEL RECURSIVE MACHINING

When I make my first set of slave "hands'' at one-fourth scale, I am going to make ten sets. I make ten sets of "hands,'' and I wire them to my original levers so they each do exactly the same thing at the same time in parallel. Now, when I am making my new devices one-quarter again as small, I let each one manufacture ten copies, so that I would have a hundred "hands'' at the 1/16th size.

Where am I going to put the million lathes that I am going to have? Why, there is nothing to it; the volume is much less than that of even one full-scale lathe. For instance, if I made a billion little lathes, each 1/4000 of the scale of a regular lathe, there are plenty of materials and space available because in the billion little ones there is less than 2 percent of the materials in one big lathe.

It doesn't cost anything for materials, you see. So I want to build a billion tiny factories, models of each other, which are manufacturing simultaneously, drilling holes, stamping parts, and so on.

Richard P. Feynman, "There's Plenty of Room at the Bottom: An Invitation to Enter a New Field of Physics," Engineering & Science, February 1960. Presented at the annual meeting of the American Physical Society, 29 December 1959. Full transcript available at <www.zyvex.com/nanotech/feynman.html>.